5.5. Complex Vector spaces. Complex Eigenvalues

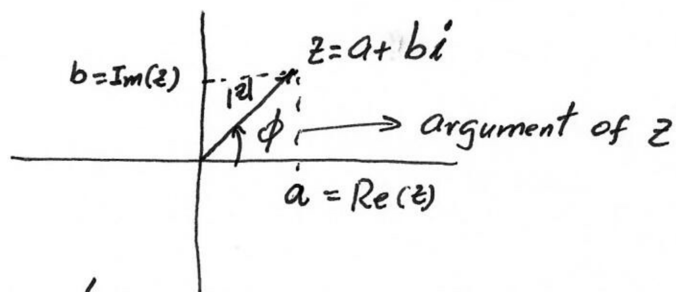
Complex Numbers:

$z = 3 + 2i \qquad Re(z) = 3, \quad Im(z) = 2$

$|z| = \sqrt{9+4} = \sqrt{13}$: Modulus

Conjugate: $\bar{z} = 3 - 2i, \qquad z\bar{z} = (3+2i)(3-2i) = 9+4 = 13 = |z|^2$

Polar form:



$Re(z) = |z|\cos\phi, \qquad Im(z) = |z|\sin\phi$

$z = |z|(\cos\phi + \sin\phi\, i)$ Polar form of z.

Complex Eigenvalues and Eigenvectors.

Consider
$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

Charact. Polyn.: $|A - \lambda I| = \mathcal{P}(\lambda) = \begin{vmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} =$

$= -(2+\lambda)(2-\lambda) + 5 = -(2^2 - \lambda^2) + 5$

$= \lambda^2 + 1$

So Eigenvalues of A Satisfy $\mathcal{P}(\lambda) = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i, \lambda_2 = -i$

$\lambda = \pm\sqrt{-1}$

We conclude that a polynomial $P(\lambda) = \lambda^2+1$ with real coefficients can have complex roots

$$\lambda_1 = i, \quad \lambda_2 = -i$$

and $P(\lambda)$ can be written as

$$P(\lambda) = \lambda^2+1 = (\lambda-i)(\lambda+i).$$

Def. (Complex Vector Space)

A vector space V in which scalars are allowed to be complex numbers is called a complex vector space

Def. $(\mathbb{C}^n)$.

$\mathbb{C}^n$ is the vector space formed by the complex vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, where $u_i \in \mathbb{C}$.

and the operations are the regular addition componentwise and scalar multiplication where the scalars are $\lambda \in \mathbb{C}$.

Prove $\mathbb{C}^n$ with the above operations is a vector space

Vectors in $\mathbb{C}^3$

$$\vec{u} = \begin{bmatrix} i \\ 3-i \\ 5+2i \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -3 \\ 2-i \end{bmatrix}.$$

$$Re(\vec{u}) = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \quad Im(\vec{u}) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\vec{u} = Re(\vec{u}) + i\,Im(\vec{u}).$$

Complex matrix 2×2.

$$A = \begin{bmatrix} 3 & 2-i \\ 5-2i & 3i \end{bmatrix}$$

$$Re(A) = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}, \quad Im(A) = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A = Re(A) + i\,Im(A)$$

Complex Conjugate of A

$$\bar{A} = \begin{bmatrix} 3 & 2+i \\ 5+2i & -3i \end{bmatrix}$$

Algebraic properties of the Complex Conjugate.

a) $\bar{\bar{\vec{u}}} = \vec{u}$, b) $\overline{k\vec{u}} = \bar{k}\,\bar{\vec{u}}$, $k \in \mathbb{C}$, $\vec{u} \in \mathbb{C}^n$, $\vec{v} \in \mathbb{C}^n$

c) $\overline{\vec{u}+\vec{v}} = \bar{\vec{u}} + \bar{\vec{v}}$

d) $\bar{\bar{A}} = A$, e) $\overline{(A^T)} = (\bar{A})^T$, f) $\overline{AB} = \bar{A}\,\bar{B}$.

where $A_{m\times k}$, $B_{k\times n}$ are complex matrices.

Thm.

If λ is an eigenvalue of $A_{n\times n}$ (real matrix), and $\vec{x}$ is a corresponding eigenvector,

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

then $\bar{\lambda}$ is an eigenvalue of A, and $\bar{\vec{x}}$ is a corresponding eigenvector. $\boxed{A\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}}$

Proof:

$$\overline{A\vec{x}} = \bar{A}\,\bar{\vec{x}} \overset{\text{A has real entries}}{=} A\,\bar{\vec{x}}$$

Also $\overline{A\vec{x}} = \overline{\lambda\vec{x}} = \bar{\lambda}\,\bar{\vec{x}}$

$\Rightarrow$ $A\bar{\vec{x}} = \bar{\lambda}\,\bar{\vec{x}}$ $\Rightarrow$ $\bar{\lambda}$ is an eigenvalue and $\bar{\vec{x}}$ is a corresponding eigenvector $\bar{\lambda}$.

Remark: $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ $\Rightarrow$ $\overline{z_1 z_2} = \overline{a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)} =$

$= a_1a_2 - b_1b_2 - i(a_1b_2 + a_2b_1) = (a_1 - ib_1)(a_2 - ib_2)$

$= \bar{z}_1\,\bar{z}_2$

back

Consider

$$A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$$

$$\det(A) = (-21)(24) = 504 \neq 0 \Rightarrow A \text{ invertible.}$$

$\Downarrow$

$\lambda = 0$ is not eigenvalue.

<u>Eigenvalues:</u>

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} -3-\lambda & 12 \\ -2 & 7-\lambda \end{vmatrix} = 0$$

or $\quad -(3+\lambda)(7-\lambda) + 24 = 0$

$$(\lambda - 7)(\lambda + 3) + 24 = 0 \iff \lambda^2 - 4\lambda - 21 + 24 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \iff (\lambda - 3)(\lambda - 1) = 0$$

or Eigenvalues are $\lambda_1 = 1, \ \lambda_2 = 3$.

To find eigenvectors corresponding to $\boxed{\lambda_1 = 1}$

We need to find $\vec{x} \neq \vec{0}$ such that

$$(A - 1I)\vec{x} = \vec{0}$$

So

$$\begin{bmatrix} -3-1 & 12 \\ -2 & 7-1 \end{bmatrix} \vec{x} = \vec{0} \iff \begin{bmatrix} -4 & 12 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

What do we expect of the columns of A?

Which is equivalent to the Linear System

$$\begin{cases} -4x_1 + 12x_2 = 0 & \text{Eq. (1)} \\ -2x_1 + 6x_2 = 0 & \text{Eq. (2)} \end{cases}$$

Obviously, Eq. (1) is a multiple of Eq. (2). Why is this expected?

So we can only use one of them to obtain x_1 and x_2

From Eq. (1) $\rightarrow$ $\boxed{-4x_1 + 12x_2 = 0}$

$$\boxed{x_1 = 3x_2}$$

Therefore,

$$\vec{x} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

So $\lambda_1 = 1$ Eeigenvalue $\vec{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Corresp. eigenvector.

Reconsider the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

We already know eigenvalues are complex

$$\boxed{\lambda_1 = i, \quad \lambda_2 = -i}$$

To find eigenvectors corresponding to $\lambda_1 = i$

We need to solve the matrix equation:

$$(A - iI)\vec{x} = \vec{0}, \qquad \vec{x} \neq \vec{0}.$$

Why are we sure that there are $\vec{x} \neq \vec{0}$ satisfying this equation?

$$(A - iI)\vec{x} = \vec{0} \iff \begin{bmatrix} -2-i & -1 \\ 5 & 2-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is also equivalent to the linear system

$$\begin{cases} -(2+i)\,x_1 - x_2 = 0 & (5.1) \\ 5\,x_1 + (2-i)\,x_2 = 0 & (5.2) \end{cases}$$

What do we expect of the columns of A and the rows of A?

Notice that

$$(2-i)(2+i) = 4-(i)^2 = 4+1 = 5.$$

then row-reducing $A - iI$

$$\begin{bmatrix} -2-i & -1 & 0 \\ 5 & 2-i & 0 \end{bmatrix} \sim \begin{bmatrix} 2+i & +1 & 0 \\ 5 & 2-i & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2+i & +1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{row2} - (2-i)(\text{row1})$$

then, $(2+i)x_1 + x_2 = 0 \Rightarrow x_2 = -(2+i)x_1$

So for $\lambda_1 = i$ eigenvectors are $= x_1 \begin{bmatrix} 1 \\ -(2+i) \end{bmatrix}$ Examples: $\begin{bmatrix} 1 \\ -2-i \end{bmatrix}, \begin{bmatrix} 2-i \\ -5 \end{bmatrix}$

Using that if λ is an eigenvalue with $\vec{v}$ assoc. eigenvector, then $A\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}$

An eigenvector for $\underline{\lambda_2 = -i} = \bar{i} = \bar{\lambda}_1$ should be

$$\bar{\vec{x}} = \begin{bmatrix} 1 \\ -2+i \end{bmatrix}$$ and any multiple (complex) of this.

Summarizing,

For $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

$P(\lambda) = \lambda^2 + 1$ is the characteristic polyn. of A.

Eeigenvalues are $\lambda_1 = i$ $\qquad \lambda_2 = -i = \bar{\lambda}_1$

Corresponding eigenvectors are $\hat{V}_1 = \begin{bmatrix} 1 \\ -2-i \end{bmatrix}$, $\quad \hat{V}_2 = \begin{bmatrix} 1 \\ -2+i \end{bmatrix} = \bar{\hat{V}}_1$

So complex eigenvalues of a real matrix A appear as conjugate pairs $\lambda, \bar{\lambda}$ and the corresponding eigenvectors are also conjugates of one another.

(Alternative to pages 5^1 and 5^2)

Notice that

$$(2-i)\times \text{Eq. }(1) \Leftrightarrow (2-i)(2+i)x_1 + (2-i)x_2 = 0$$

or $$5x_1 + (2-i)x_2 = 0$$

Which is exactly Eq. (5.2)

So Eq (5.2) is a multiple of Eq. (5.1) as expected.

From this equation:

$$x_1 = \left(-\frac{2}{5} + \frac{1}{5}i\right)x_2$$

or $$\vec{x} = x_2 \begin{bmatrix} -\frac{2}{5}+\frac{1}{5}i \\ 1 \end{bmatrix}$$ eigenvectors corresponding to $\lambda_1 = i$.

The Set $B = \left\{ \begin{bmatrix} -\frac{2}{5}+\frac{1}{5}i \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_1 = i$.

We also know that $\boxed{\lambda_2 = -i}$ is an eigenvalue of A. We could proceed as above to find its corresponding eigenvectors, $\mathbf{z}$.

However using thm in page 4, we know

$\lambda_2 = \overline{\lambda_1} = \overline{i} = -i$ and $\vec{z} = \overline{\vec{x}} = \begin{bmatrix} -\frac{2}{5}-\frac{1}{5}i \\ 1 \end{bmatrix}$ is a corresponding eigenvector to λ_2

Summarizing

For $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

$\mathcal{P}(\lambda) = \lambda^2 + 1$ is the Characteristic polynomial of A

Eigenvalues: $\lambda_1 = i$ and $\lambda_2 = \bar{\lambda} = -i$

$\downarrow$

Eigenvectors:

$$\vec{x} = \begin{bmatrix} -\frac{2}{5} + \frac{1}{5} i \\ 1 \end{bmatrix} \qquad \vec{z} = \bar{\vec{x}} = \begin{bmatrix} -\frac{2}{5} - \frac{1}{5} i \\ 1 \end{bmatrix}$$

Notice

$$Re(\vec{x}) = \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}, \qquad Im(\vec{x}) = \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}$$

$$Re(\vec{z}) = \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}, \qquad Im(z) = \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix}$$

Theorem 9.1

The eigenvalues of the real matrix

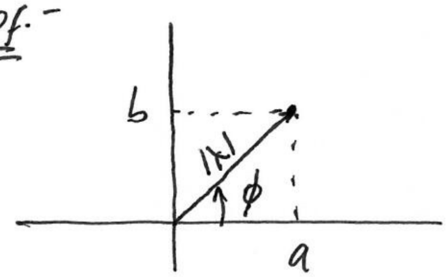
$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

are $\lambda = a \pm ib$. If a or b are not both zero, then this matrix can be factored as

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \qquad (8.1)$$

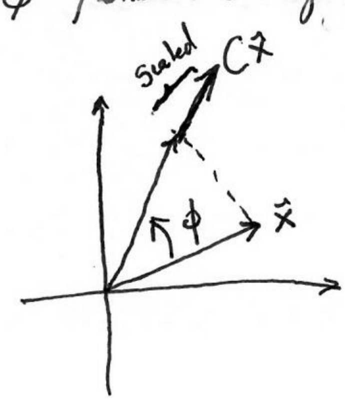
where ϕ is the angle from the positive x-axis to the ray that joins the origin to the point (a,b).

Proof.-



$$\cos\phi = \frac{a}{|\lambda|}, \quad \sin\phi = \frac{b}{|\lambda|} \qquad (8.2)$$

Equ. (8.1) means that $C\vec{x}$ can be viewed as a rotation of vector $\vec{x}$ by an angle ϕ followed by a scaling with factor $|\lambda|$.



$$g(\lambda) = |C-\lambda I| = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2+b^2$$

So $g(\lambda)=0 \iff (a-\lambda)^2=-b^2 \Rightarrow \boxed{\lambda = a\pm ib.}$

From (8.2), it follows

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda|\cos\phi & -|\lambda|\sin\phi \\ |\lambda|\sin\phi & |\lambda|\cos\phi \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

So $C\vec{x}$, means a rotation by a ϕ angle followed by a scaling through a factor $|\lambda|$.

Thm 9.2

1) $A_{2\times 2}$ real matrix

2) $\lambda = a\pm ib$ Complex eigenvalues $(b\neq 0)$

3) $\vec{x}$ an eigenvector of A Corresponding to $\lambda = a-ib$

4) $P = [Re(\vec{x}) \quad Im(\vec{x})]$

then a) P is invertible.

and b) $$A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}.$$

In our previous example

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} \qquad \lambda = \pm i$$

For $\lambda = -i$ Eigenvector $\vec{x} = \begin{bmatrix} -\frac{2}{5} - \frac{1}{5} i \\ 1 \end{bmatrix}$

Then

$$P = \left[Re(\vec{x}) \;\; Im(\vec{x}) \right] = \begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ 1 & 0 \end{bmatrix}$$

So according to thm 9.2

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} = \overset{P}{\begin{bmatrix} -\frac{2}{5} & -\frac{1}{5} \\ 1 & 0 \end{bmatrix}} \overset{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix}}$$

Verify this!

Geometrical Interpretation

Symmetric Matrices Have Real Eigenvalues

Our next result, which is concerned with the eigenvalues of real symmetric matrices, is important in a wide variety of applications. The key to its proof is to think of a real symmetric matrix as a complex matrix whose entries have an imaginary part of zero.

THEOREM 5.3.6 *If A is a real symmetric matrix, then A has real eigenvalues.*

Proof Suppose that λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, where we allow for the possibility that λ is complex and $\mathbf{x}$ is in C^n. Thus,

$$A\mathbf{x} = \lambda\mathbf{x}$$

where $\mathbf{x} \neq \mathbf{0}$. If we multiply both sides of this equation by $\overline{\mathbf{x}}^T$ and use the fact that

$$\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T(\lambda\mathbf{x}) = \lambda(\overline{\mathbf{x}}^T\mathbf{x}) = \lambda(\mathbf{x}\cdot\mathbf{x}) = \lambda\|\mathbf{x}\|^2$$

then we obtain

$$\lambda = \frac{\overline{\mathbf{x}}^T A\mathbf{x}}{\|\mathbf{x}\|^2}$$

Since the denominator in this expression is real, we can prove that λ is real by showing that

$$\overline{\overline{\mathbf{x}}^T A\mathbf{x}} = \overline{\mathbf{x}}^T A\mathbf{x} \tag{14}$$

But, A is symmetric and has real entries, so it follows from the second equality in (14) and properties of the conjugate that

$$\overline{\overline{\mathbf{x}}^T A\mathbf{x}} = \overline{\overline{\mathbf{x}}}^T\,\overline{A\mathbf{x}} = \mathbf{x}^T\overline{A\mathbf{x}} = (\overline{A\mathbf{x}})^T\mathbf{x} = (\overline{A}\,\overline{\mathbf{x}})^T\mathbf{x} = (A\overline{\mathbf{x}})^T\mathbf{x} = \overline{\mathbf{x}}^T A^T\mathbf{x} = \overline{\mathbf{x}}^T A\mathbf{x} \quad ◀$$

A Geometric Interpretation of Complex Eigenvalues

The following theorem is the key to understanding the geometric significance of complex eigenvalues of real 2×2 matrices.

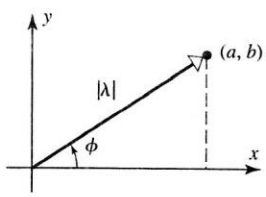
THEOREM 5.3.7 *The eigenvalues of the real matrix*

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{15}$$

are $\lambda = a \pm bi$. If a and b are not both zero, then this matrix can be factored as

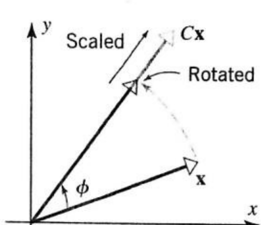
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{16}$$

where ϕ is the angle from the positive x-axis to the ray that joins the origin to the point (a, b) (Figure 5.3.2).



▲ **Figure 5.3.2**

Geometrically, this theorem states that multiplication by a matrix of form (15) can be viewed as a rotation through the angle ϕ followed by a scaling with factor $|\lambda|$ (Figure 5.3.3).



▲ **Figure 5.3.3**

Proof The characteristic equation of C is $(\lambda - a)^2 + b^2 = 0$ (verify), from which it follows that the eigenvalues of C are $\lambda = a \pm bi$. Assuming that a and b are not both zero, let ϕ be the angle from the positive x-axis to the ray that joins the origin to the point (a, b). The angle ϕ is an argument of the eigenvalue $\lambda = a + bi$, so we see from Figure 5.3.2 that

$$a = |\lambda|\cos\phi \quad \text{and} \quad b = |\lambda|\sin\phi$$

Power Sequences There are many problems in which one is interested in how successive applications of a matrix transformation affect a specific vector. For example, if A is the standard matrix for an operator on R^n and $\mathbf{x}_0$ is some fixed vector in R^n, then one might be interested in the behavior of the power sequence

$$\mathbf{x}_0, \quad A\mathbf{x}_0, \quad A^2\mathbf{x}_0, \ldots, \quad A^k\mathbf{x}_0, \ldots$$

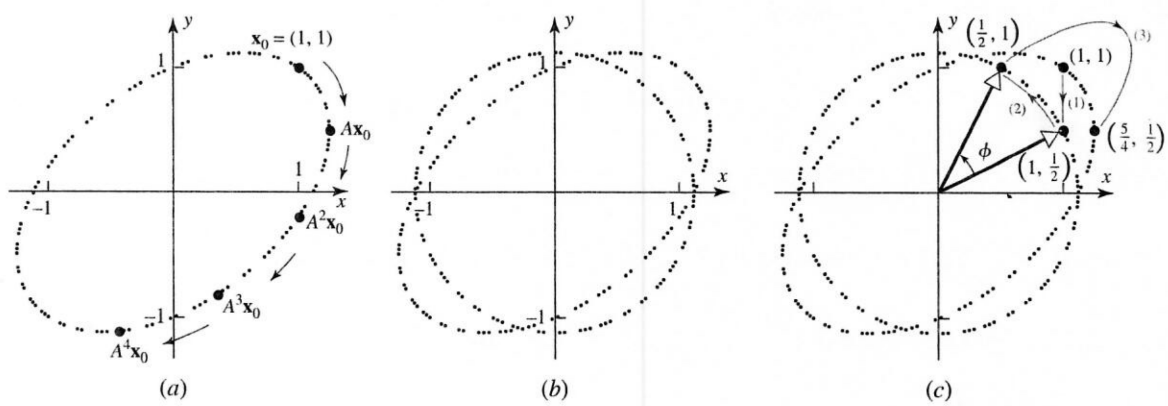
For example, if

$$A = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{3}{5} & \frac{11}{10} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then with the help of a computer or calculator one can show that the first four terms in the power sequence are

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A\mathbf{x}_0 = \begin{bmatrix} 1.25 \\ 0.5 \end{bmatrix}, \quad A^2\mathbf{x}_0 = \begin{bmatrix} 1.0 \\ -0.2 \end{bmatrix}, \quad A^3\mathbf{x}_0 = \begin{bmatrix} 0.35 \\ -0.82 \end{bmatrix}$$

With the help of MATLAB or a computer algebra system one can show that if the first 100 terms are plotted as ordered pairs (x, y), then the points move along the elliptical path shown in Figure 5.3.4a.



▲ **Figure 5.3.4**

To understand why the points move along an elliptical path, we will need to examine the eigenvalues and eigenvectors of A. We leave it for you to show that the eigenvalues of A are $\lambda = \frac{4}{5} \pm \frac{3}{5}i$ and that the corresponding eigenvectors are

$$\lambda_1 = \tfrac{4}{5} - \tfrac{3}{5}i: \quad \mathbf{v}_1 = \left(\tfrac{1}{2} + i, 1\right) \quad \text{and} \quad \lambda_2 = \tfrac{4}{5} + \tfrac{3}{5}i: \quad \mathbf{v}_2 = \left(\tfrac{1}{2} - i, 1\right)$$

If we take $\lambda = \lambda_1 = \frac{4}{5} - \frac{3}{5}i$ and $\mathbf{x} = \mathbf{v}_1 = \left(\frac{1}{2} + i, 1\right)$ in (17) and use the fact that $|\lambda| = 1$, then we obtain the factorization

$$\underset{A}{\begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{3}{5} & \frac{11}{10} \end{bmatrix}} = \underset{P}{\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}} \underset{R_\phi}{\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}} \tag{19}$$

where R_ϕ is a rotation about the origin through the angle ϕ whose tangent is

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{3/5}{4/5} = \frac{3}{4} \qquad \left(\phi = \tan^{-1}\tfrac{3}{4} \approx 36.9^\circ\right)$$

The matrix P in (19) is the transition matrix from the basis

$$B = \{\text{Re}(\mathbf{x}), \text{Im}(\mathbf{x})\} = \left\{\left(\tfrac{1}{2}, 1\right), (1, 0)\right\}$$